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Black Hole Complementary Principle and Noncommutative Membrane*

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Abstract *In the spirit of black hole complementary principle, we have found the noncommutative membrane of Scharzchild black holes. In this paper we extend our results to Kerr black hole and see the same story. Also we make a conjecture that spacetimes are noncommutative on the stretched membrane of the more general Kerr–Newman black hole.*

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1 Introduction

To solve the black hole information loss problem,[1] Susskind found the black hole complementary principle,[2] which says that there is no such super-observer who can see both the inside and outside of black holes, and what the inside and outside observers see are complementary. From this principle what the outside observer sees is self-consistent, and there is no black hole information loss for him because of the black hole information red-shift on the horizon and we can take the black hole as a physical object.

For the outside observer who is near the horizon, the black hole behaves like a hot stretched membrane. This stretched membrane is much different from its original meaning, which is just a classical fictitious construct.[3] In fact for this observer, all the quantum information of the black hole is stored on the hot membrane and we can get some insight from the membrane.[4]

In our previous paper[5] we have found the unusual dispersion relation $E = m^2/k$ on the stretched membrane, which indicates the noncommutative spacetimes on the membrane for the Schwarzschild black hole. In the spirit of black hole complementary principle we extend our results to Kerr black hole and see the same story. Also we make a conjecture that spacetimes are noncommutative on the stretched membrane of the more general Kerr–Newman black hole.

In this paper we will first give a review of our results for the Schwarzschild black hole, and the extended results for Kerr black holes will be given next. Finally we will show some arguments to make our conjecture.

2 Noncommutative Membrane of Scharzchild Black Holes

In this section we will give a brief review of our previous results for Schwarzschild black holes. The four-dimensional black hole with the Schwarzschild metric takes the form

$$\mathrm{d}s^2 = -\Big(1-\frac{r_0}{r}\Big)\mathrm{d}t^2 + \Big(1-\frac{r_0}{r}\Big)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2\,. \quad (1)$$

For the infinite observer the Hawking temperature and the Bekenstein–Hawking entropy are

$$T_H = \frac{1}{4\pi r_0}\,,\quad S = \frac{A}{4l_p^2}\,,\quad l_p^2 = G\,, \quad (2)$$

where A is the area of the horizon. For the observer near the horizon the black hole looks like a hot stretched (2+1)-dimensional membrane and the Hawking temperature will be blue-shifted,

$$T = g_{00}^{-1/2}(r_s)T_H\,. \quad (3)$$

If we attribute the black hole mass completely to the stretched horizon, since the mass is also blue-shifted, the mass density on the stretched horizon is

$$\rho = Mg_{00}^{-1/2}(r_s)\frac{1}{A}\,. \quad (4)$$

The density of entropy is simply

$$\sigma = \frac{S}{A} = \frac{1}{4l_p^2}\,. \quad (5)$$

For the Schwarzschild black hole the Smarr's mass formula[6] is

$$M = 2T_H S\,. \quad (6)$$

So from the above relations we can easily get

$$\rho = 2T\sigma\,. \quad (7)$$

As was stated in our previous paper the above formulas mean that the energy per degree of freedom is $2T$. It is a strange result because the energy per degree is $T/2$ for the one-dimensional relativistic gas, and for a d-dimensional relativistic gas, we have $\rho/\sigma = (d/(d+1))T$. Thus, we can never hope to get the result $\rho/\sigma = 2T$ from any known gas.

Let us assume that on the membrane there is a perfect fluid. We shall for now imagine that its temperature, energy density, and entropy density can be changed, and

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then we can apply the first law of thermodynamics to the membrane to gain some insight into this fluid:

$$\mathrm{d}(\rho A) = T\mathrm{d}(\sigma A) - p\mathrm{d}A\,. \tag{8}$$

So we get

$$A(\mathrm{d}\rho - T\mathrm{d}\sigma) + (\rho - T\sigma + P)\mathrm{d}A = 0\,. \tag{9}$$

Because of the symmetry of the Schwarzschild metric, the thermodynamic quantities measured by the observer near the horizon should be independent of A. Therefore we obtain

$$\rho = T\sigma - p\,, \quad \mathrm{d}\rho = T\mathrm{d}\sigma\,. \tag{10}$$

From the first relation and Eq. (7) we get

$$p = -T\sigma = -\frac{\rho}{2}\,. \tag{11}$$

Thus the pressure is negative, which may be responsible for the stability of the membrane under self-gravity. From Eq. (7) and the second relation we get

$$\rho = 2cT^{-1}\,, \quad \sigma = cT^{-2}\,. \tag{12}$$

Obviously the above relation (12) is unusual.

Next we try to use statistical mechanics to gain some insight into the microscopic nature of the membrane fluid. All thermodynamic quantities can be obtained from the free energy density f,

$$\rho = \partial_\beta f\,, \quad \sigma = \beta\rho - f\,. \tag{13}$$

To obtain Eq. (12) we should require that f scales with T as $f \sim \beta^2$. For the noninteracting relativistic gas we have

$$f = \int_0^\infty F(\beta k) k\mathrm{d}k = \int_0^\infty \mp \ln[1 \pm \exp(-\beta k)] k\mathrm{d}k\,, \tag{14}$$

where F as a function of βk depends on the nature of the constituents, bosons or fermions and the measure $k\mathrm{d}k$ in the momentum space is standard for a two-dimensional surface. In order to have $f \sim T^{-2}$, one can either replace the measure $k\mathrm{d}k$ by $k^{-3}\mathrm{d}k$, or to change the dispersion relation. We have no reason to change the measure (a consequence of quantum mechanics). Thus, we need to change the dispersion relation $E = E(k)$ in $F(\beta E(k))$. It is easy to see that if the dispersion relation satisfies

$$E = \frac{m^2}{k}\,, \tag{15}$$

we get the desired result $f \sim T^{-2}$. Thus in conclusion, the following are forced upon us,

$$f = \int_0^\infty F\Big(\frac{\beta m^2}{k}\Big) k\mathrm{d}k = am^4\beta^2 \tag{16}$$

with

$$a = \int_0^\infty F(k^{-1}) k\mathrm{d}k\,. \tag{17}$$

We should remember that the usual quantum mechanics is still valid on membrane, so momentum is still conjugate to space, and time is conjugate to energy, thus $E \sim 1/\Delta T$ and $k \sim 1/\Delta X$. Thus the unusual dispersion relation $E = m^2/k$ naturally implies

$$\Delta T \Delta X \sim m^{-2}\,. \tag{18}$$

Therefore we have found that the spacetimes on the stretched membrane are noncommutative.[7] In our previous paper another interpretation was given that the constituents of the membrane fluid are microscopic black holes. But as we will show in the next section, this picture may be wrong because all the thermodynamic quantities on the stretched membrane depend on the local observer and have just local meaning, and so the global interpretation should be meaningless that the constituents of the membrane fluid are microscopic black holes.

3 Noncommutative Membrane of Kerr Black Holes

Intuitively we can guess that for the local property of the stretched membrane there should be no difference between the Schwarzschild and the Kerr black holes if the observer comoves with the horizon. In the following we can see this point.

The Kerr black hole metric in Boyer–Lindquist coordinate $(x^0 = t, x^i = r, \theta^\dagger, \phi^\dagger)$, is given by

$$\mathrm{d}s^2 = -\Big(\rho^2\frac{\Delta}{\Sigma^2}\Big)\mathrm{d}t^2 + g_{jk}(\mathrm{d}x^j + \beta^k\mathrm{d}t)(\mathrm{d}x^k + \beta^k\mathrm{d}t)\,, \tag{19}$$

where

$$g_{jk}\mathrm{d}x^j\mathrm{d}x^k = \frac{\rho^2}{\Delta}\mathrm{d}r^2 = \rho^2\mathrm{d}\theta^{\dagger 2} + \Big(\frac{\Sigma\sin\theta^\dagger}{\rho}\Big)^2\mathrm{d}\phi^{\dagger 2}\,, \tag{20}$$

$$\Delta \equiv r^2 + a^2 - 2Mr\,, \quad \rho^2 \equiv r^2 + a^2\cos^2\theta^\dagger\,,$$
$$\Sigma^2 \equiv (r^2 + a^2)^2 - a^2\Delta\sin^2\theta^\dagger\,, \tag{21}$$

and

$$\beta^{\phi^\dagger} = -\frac{2Mar}{\Sigma^2}\,, \quad \beta^r = \beta^{\theta^\dagger} = 0\,. \tag{22}$$

The event horizon is at

$$r_H = r_+\,, \quad r_\pm \equiv M \pm (M^2 - a^2)^{1/2}\,, \tag{23}$$

where $r_\pm$ are roots of $\Delta = 0$.

Since the stretched horizon is very close to r_H, we can arrive at a more convenient set of coordinate,

$$\alpha \equiv \rho\frac{\Delta^{1/2}}{\Sigma} = \Big[\frac{1 - a^2\sin^2\theta^\dagger/2Mr_H}{Mr_H/(r_H - M)}\Big]^{1/2}(r - r_H)^{1/2} + O[(r - r_H)^{3/2}]\,, \tag{24}$$

which vanishes at the event horizon and increases outward. Next we introduce the new angular coordinates θ' and ϕ' by

$$\theta' = \theta^\dagger - \frac{\rho^2_{H,\theta^\dagger}}{\kappa^2\rho_H^4}\alpha^2\,, \quad \phi' = \phi^\dagger - \Omega_H t\,, \tag{25}$$

where Ω_H is the angular velocity of the horizon with $\Omega_H = a/2Mr_H$ and $\rho_H^2 = r_H^2 = a^2\cos^2\theta^\dagger$ is the value of ρ^2 at the horizon. In terms of these coordinates the spacetime metric takes the form

$$\mathrm{d}s^2 = -\alpha^2\mathrm{d}t^2 + \kappa^{-2}\mathrm{d}\alpha^2 + \rho_H^2\mathrm{d}\theta^2 + \tilde{\omega}_H^2(\mathrm{d}\phi' + \beta^{\theta'}\mathrm{d}t)^2\,, \tag{26}$$

$$\kappa \equiv \frac{r_H - M}{2MrH}\,, \quad \tilde{\omega}_H^2 \equiv \frac{(2MrH)^2}{\rho_H^2}\sin^2\theta'\,, \tag{27}$$

$$\beta^{\theta'} \equiv \frac{\alpha^2 a}{\kappa(2MrH)^2\rho_{_H}^2}\left[\rho_{_H}^2 r_{_H} + M(r_{_H}^2 - a^2\cos^2\theta')\right]. \quad (28)$$

One advantage of the new angular variables is that they comove with the horizon. Because we focus our attention on the stretched horizon ($\alpha \ll 1$), which is sufficiently close to the event horizon then in Eq. (26) the $g_{t\phi'}$ term, which is $O(a\alpha^2\sin^2\theta')$, may be ignored and the Kerr geometry may be approximated as

$$\mathrm{d}s^2 = -\alpha^2\mathrm{d}t^2 + \kappa^{-2}\mathrm{d}\alpha^2 + \rho_{_H}^2\mathrm{d}\theta^2 + \tilde{\omega}_{_H}^2\mathrm{d}\phi'^2 . \quad (29)$$

In the membrane paradigm the surface at $\alpha = \alpha_{_H} \ll 1$ is called the "stretched horzion", which is timelike. The physical variables such as the temperature T_s, mass density ρ_o on the two-dimensional membrane are blue-shifted, which are given by

$$\rho_o = \frac{1}{\alpha_{_H}}\rho = \frac{1}{\alpha_{_H}}\frac{M}{A}, \quad T_s = \frac{1}{\alpha_{_H}}T_H = \frac{1}{\alpha_{_H}}\frac{\kappa}{2\pi}. \quad (30)$$

Also on the membrane, the density of entropy is

$$\sigma = \frac{S}{A_s}, \quad S = \frac{A}{4}, \quad A_s \simeq A . \quad (31)$$

Obviously T_s and ρ_o have only local meaning that depends on θ' and r. Also we should realize that in the above definition of the mass density of the stretched membrane, the observer is obscure and we just give a blue-shift factor before the physical quantities measured by the infinite observer. From the new coordinates (25) and the metric (29) we can learn that the observer co-rotates with the horizon. So the above definition of the mass density is just a formal one and its physical meaning is obscure. To get the physical mass density ρ_s measured by the observer who is near the horizon and comoves with it, we should remove the contribution of the angular momentum to the formal one ρ_o. Also we should notice that the local temperature T_s is physical because it is a macroscopic thermodynamic quantity and should be independent of the angular momentum.

From Smarr's mass formula for Kerr black hole:

$$M = \frac{A\kappa}{4\pi} + 2\Omega_H J , \quad (32)$$

where

$$\kappa = \frac{r_+ - r_-}{4Mr_+}, \quad A = 4\pi(r_{_H}^2 + a^2),$$
$$\Omega_H = \frac{a}{2Mr_+}, \quad (33)$$

we can obtain

$$\rho_o = 2T_s\Big(\sigma + \frac{2\pi\Omega_H J}{\kappa A}\Big). \quad (34)$$

As was argued in the above, we should remove the contribution of the angular momentum to the formal one ρ_o. So the physical mass density ρ_s measured by the observer comoving with the horizon should be

$$\rho_s = \rho_o - 2T_s\frac{2\pi\Omega_H J}{\kappa A} = 2T_s\sigma . \quad (35)$$

As we see, for the local physical quantities relation (35) there is no difference between the Schwarzschild and the Kerr black holes.

Now we apply the first law of thermodynamics to a small pitch ΔA of the hot stretched membrane

$$\mathrm{d}(\rho_s\Delta A) = T_s\mathrm{d}(\sigma\Delta A) - p\mathrm{d}(\Delta A). \quad (36)$$

Therefore

$$\Delta A(\mathrm{d}\rho_s - T_s\mathrm{d}\sigma) + \mathrm{d}(\Delta A)(\rho_s - T_s\sigma + p) = 0 . \quad (37)$$

For the second term

$$\mathrm{d}(\Delta A) = \mathrm{d}\Big(\Delta\int_{\Delta A}\sqrt{g_{\phi\phi}g_{\theta\theta}}\,\mathrm{d}\phi\mathrm{d}\theta\Big)$$
$$\simeq \Delta(\sqrt{g_{\phi\phi}g_{\theta\theta}})\mathrm{d}\phi\mathrm{d}\theta , \quad (38)$$

since ρ_s and σ are independent of ϕ we can get

$$\rho_s = T_s\sigma - p , \quad \mathrm{d}\rho_s = T_s\mathrm{d}\sigma . \quad (39)$$

From the second relation and Eq. (35) we get

$$\rho_s = 2cT_s^{-1}, \quad \sigma = cT_s^{-2}. \quad (40)$$

From the first relation and Eq. (40) we get

$$p = -cT_s^{-1}. \quad (41)$$

Following the Schwarzschild black hole case, we can get the same unusual dispersion relation $E = m^2/k$, which implies the interpretation that spacetimes on the stretched membrane is noncommutative.

4 Noncommutative Membrane of Kerr–Newman Black Holes

As the Kerr black hole case, the local property of the stretched membrane of Kerr–Newman black hole should be the same to the Newman black hole. There are some difficulties in directly obtaining the expected result. But we can still make a conjecture that spacetimes on the stretched membrane of the Newman black hole are noncommutative.

For Newman black hole, the metric is

$$\mathrm{d}s^2 = -\Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big)\mathrm{d}t^2 + \Big(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\Big)^{-1}\mathrm{d}r^2$$
$$+ r^2\mathrm{d}\Omega^2 . \quad (42)$$

The local physical quantities of Newman black holes are defined as the Schwarzschild black hole. After applying the Smarr's mass formulas for Newman black holes and the first law of thermodynamics to the stretched membrane, we can easily get

$$\rho_s = 2cT_s^{-1}, \quad \sigma = cT_s^{-2} - 2\pi\Phi_H Q(\kappa A)^{-1}, \quad (43)$$

where Φ_H is the co-rotating electric potential on the horizon. In the second relation there is an additional term from which we cannot get the expected dispersion relation by making analysis of the form of the free energy density f in Eq. (14).

But we notice that the free energy density (14) is adapted to the noninteracting relativistic gas. For the stretched membrane of Newman black holes, the microscopic constituent can interact each other because of electric charges, and the free energy density f should have a different form. So we can make a conjecture that it is a general result for all kinds of black holes that the spacetimes on the stretched membrane is noncommutative.

5 Conclusion

Noncommutative spacetimes should be a natural result of quantum gravity, especially string theory. In the past years, Noncommutative spacetimes has found its role in cosmology,[8] so we believe it can also play an important role in other fields when quantum gravity effects must be considered.

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